5. MODELLING OF NOISE

Motivation:

- analytical treatment can be very complex
- having only physical, intuitive picture
- verifying theoretical solutions
- testing with real, physical noise sources

Simulation is very powerful, however neglects several real facts: always check with measurements, if possible.

5.1. Analog simulation

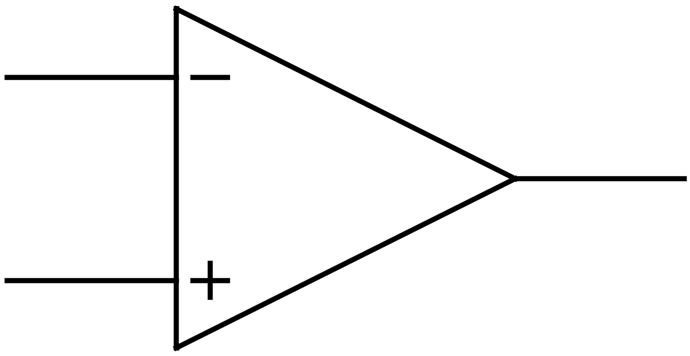
Analog computation: uses analog operational elements

- quantities -> voltage, current

- equations -> realized by electronics

5.1.1. Operational units

5.1.1.1. Operational amplifier

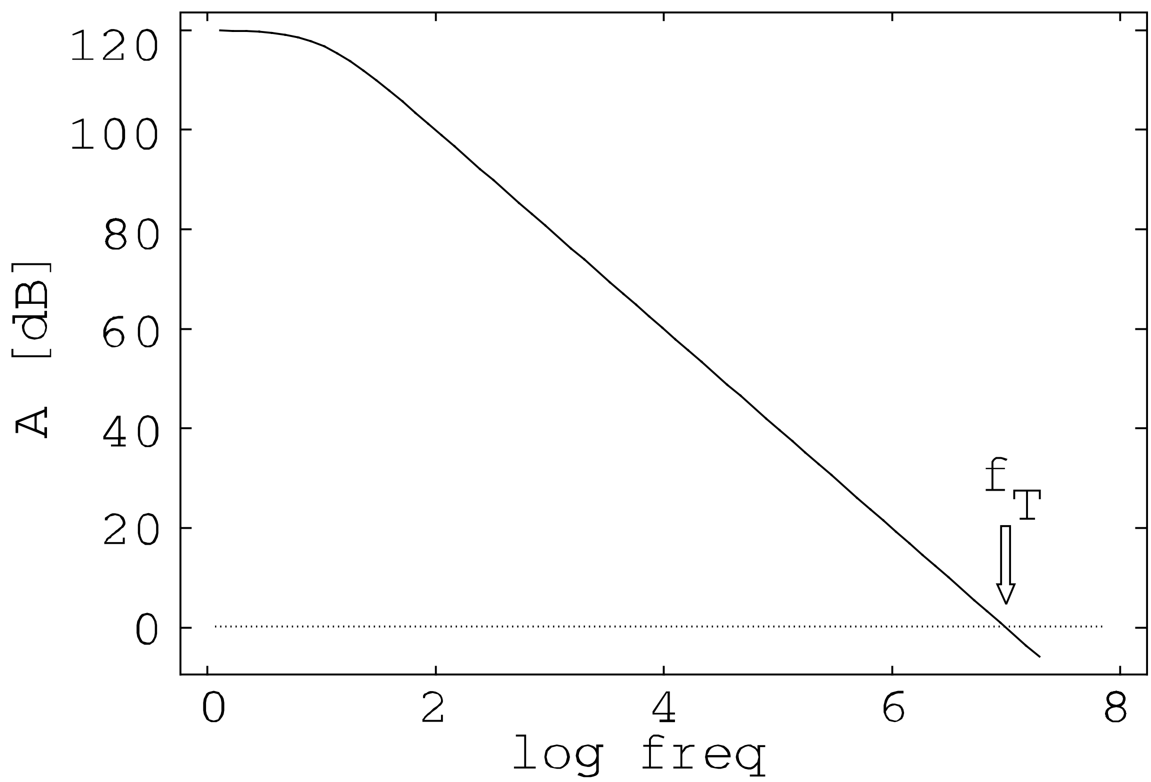


Ideal case:

- $V_{out}=A(V_+-V_-)$, $A=\infty$
- input resistance = ∞
- input currents = 0

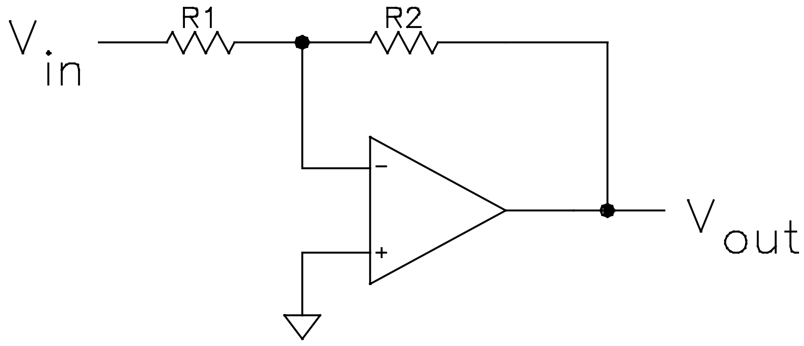
Real:

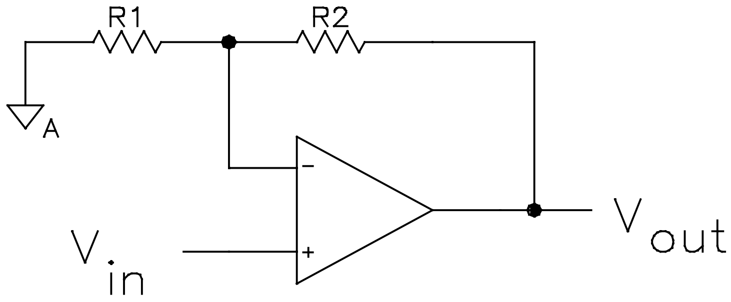
- A(f), at DC: $10^4..10^7$
- offset voltage (<mV), input currents (nA), input resistance (>MΩ)
- temperature dependence, etc.



120
100
80
60
40
20
0
A [dB]
0
2
4
6
8
log freq
f_T

5.1.1.2. Amplification





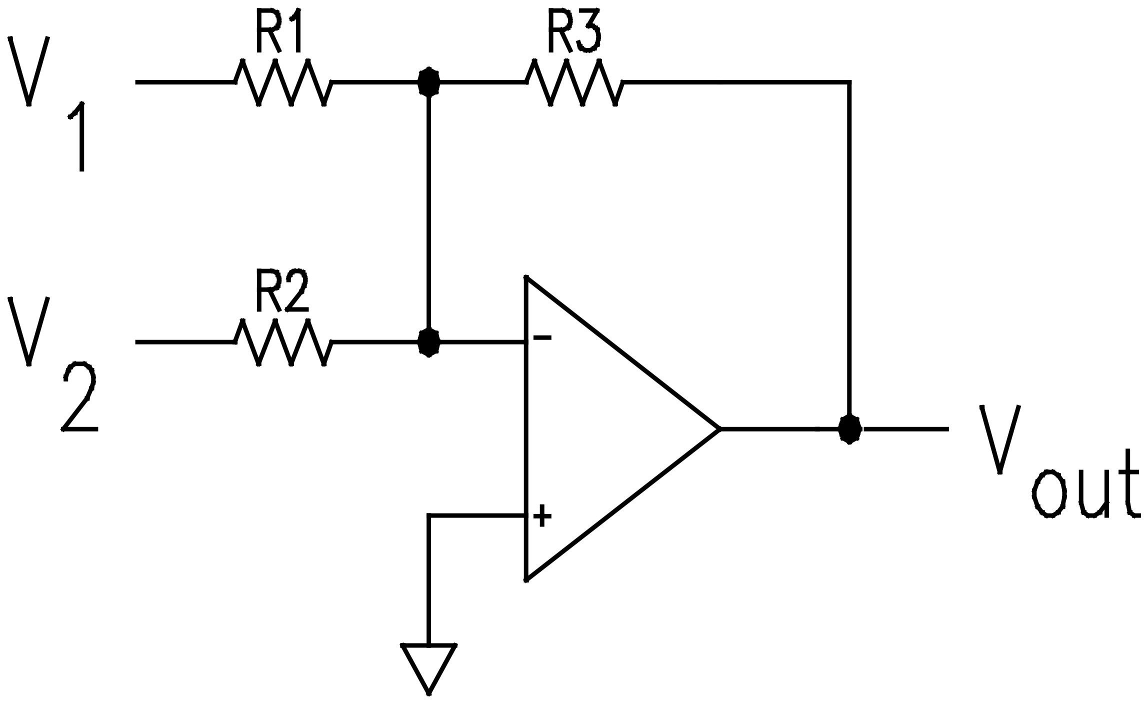
- inverting:

  $V_{out}/V_{in}=G=-R_2/R_1$

- non-inverting

  $V_{out}/V_{in}=G=1+R_2/R_1$

5.1.1.3. Addition

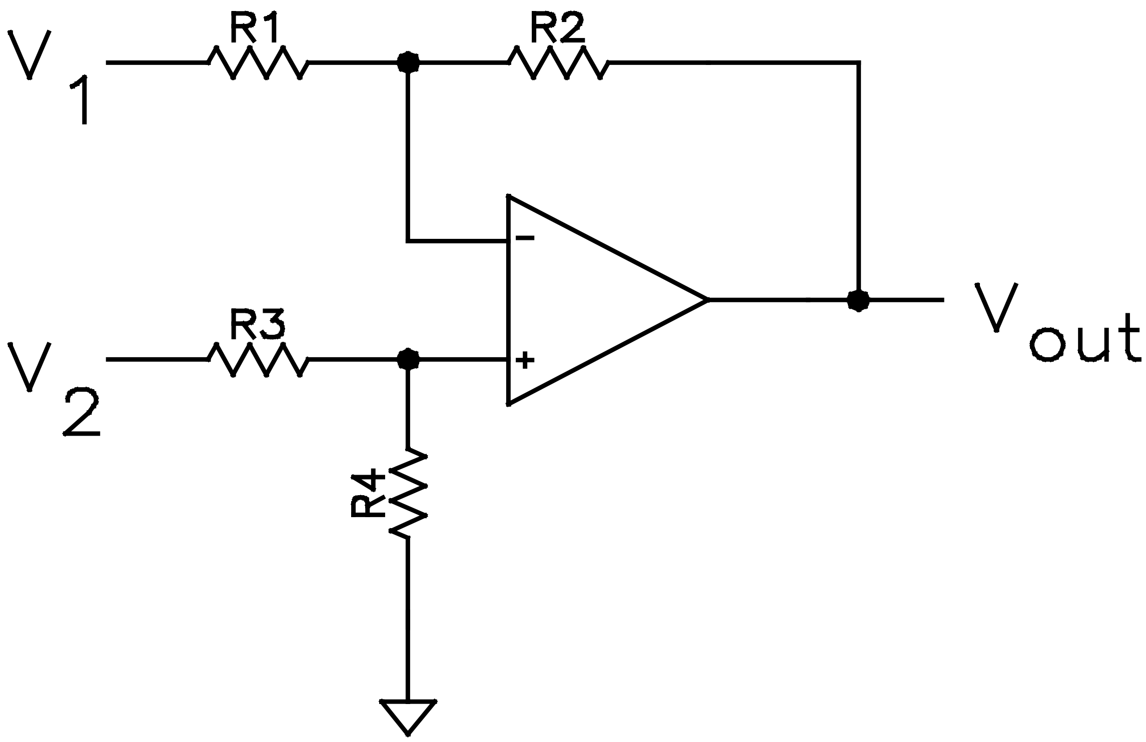


$$V_{out} = -\left(\frac{R_3}{R_1} V_1 + \frac{R_3}{R_2} V_2 \right)$$

If R1=R2=R3

$$V_{out} = -\left(V_1 + V_2 \right)$$

5.1.1.4. Subtraction

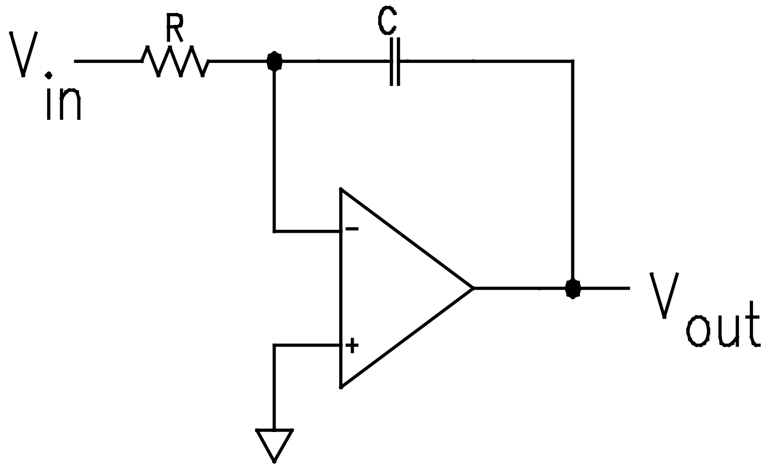


$$V_{out}=\left(\frac{R_1+R_2}{R_3+R_4}\right)\frac{R_4}{R_1}V_2-\frac{R_2}{R_1}V_1$$

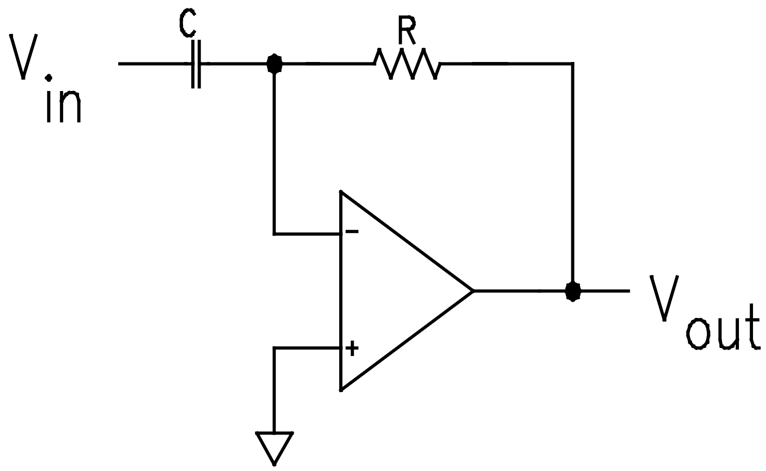
if R1=R2=R3=R4:

$$V_{out}=V_2-V_1$$

5.1.1.5. Integration, differentation

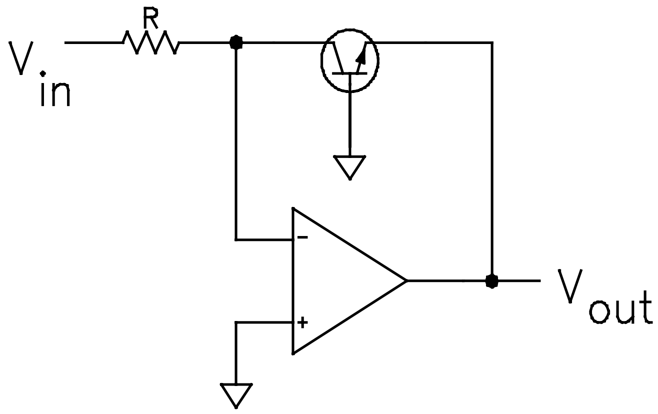


$$V_{out} = -\frac{1}{RC}\int V_{in}(t)\,dt$$

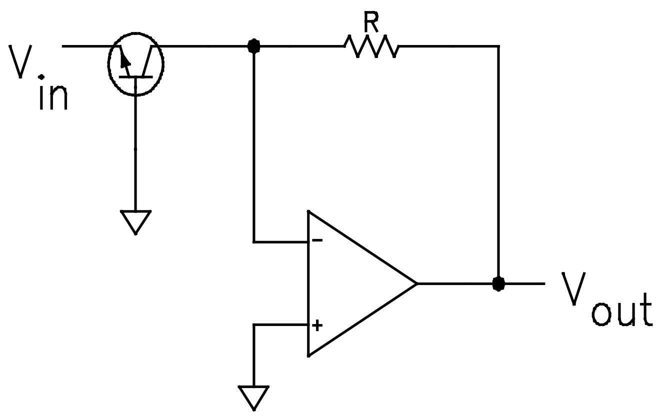


$$V_{out} = -RC\frac{dV_{in}(t)}{dt}$$

5.1.1.6. Log, exp functions



$$V_{out} = -\frac{kT}{q} \ln \frac{V_{in}}{I_o R}$$



$$V_{out} = I_o R e^{-q \frac{V_{in}}{kT}}$$

5.1.1.7. Multiplication, square root

Possibilities:

- using exp/log functions
- preferred: special circuits

5.1.1.8. Function approximation

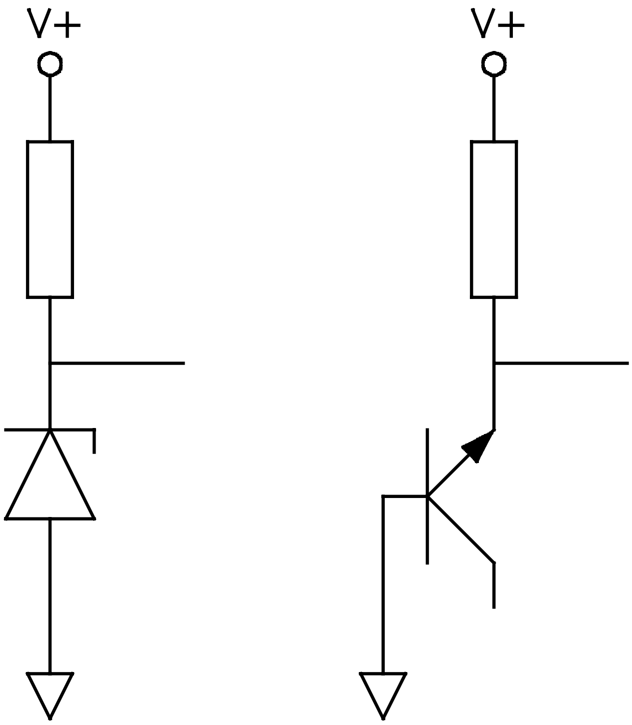
Special functions:

- build from previous operations
- approximate with Taylor series
- special circuits

5.1.2. Noise generation methods

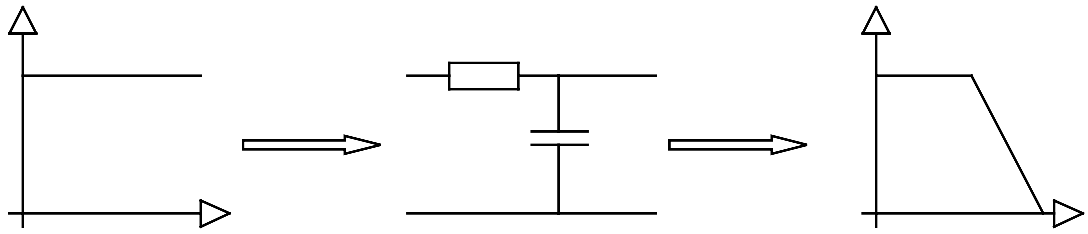
5.1.2.1. White noise

- cooled or warmed resitor
- biased zener and amplification
- biased transistor EB junction
- high frequency cutoff always present



5.1.2.2. Lorentzian noise

Filtered white noise:



Corner frequency: $f=1/2\pi RC$

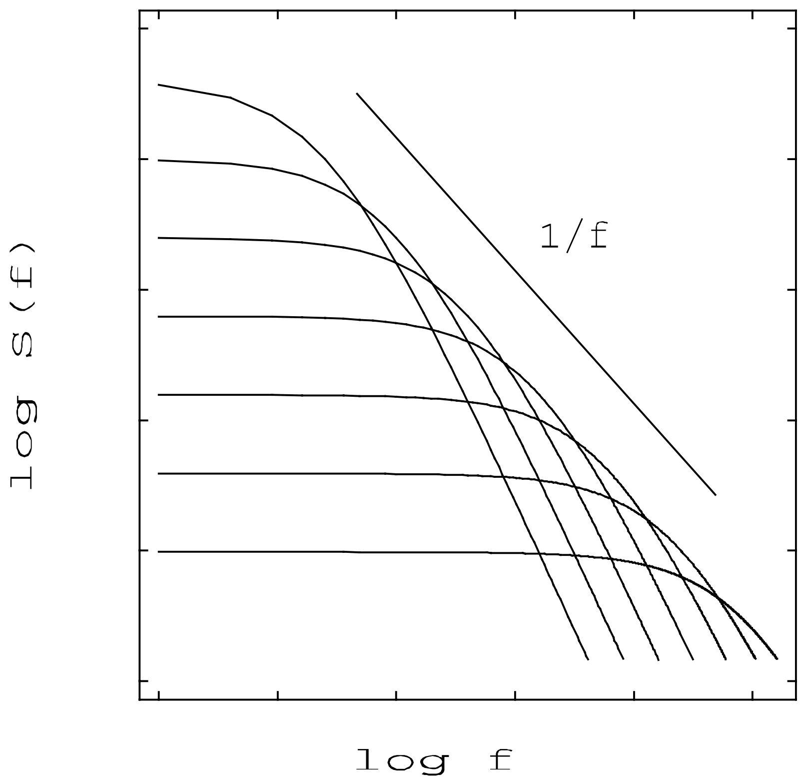
Correlation time: $\tau=RC$

5.1.2.3. $1/f^2$ noise

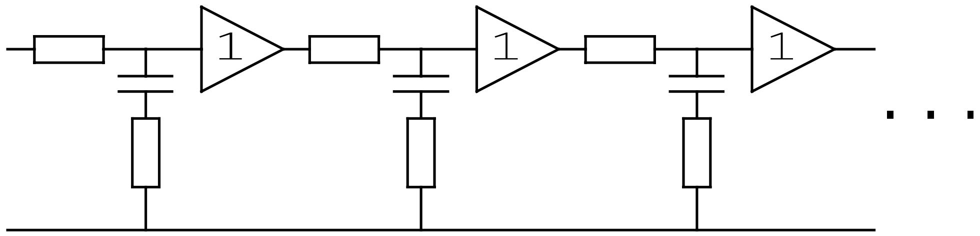
- integrated white noise
- to avoid divergence: low frequency cutoff

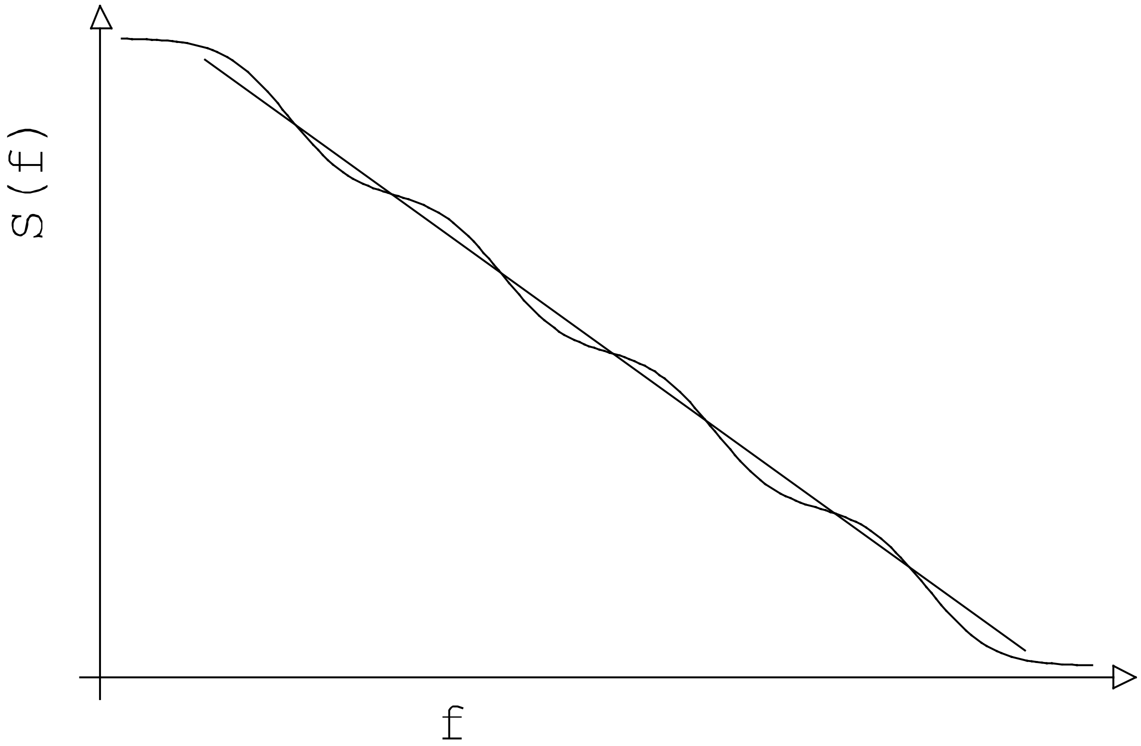
5.1.2.4. 1/f noise

- natural amplified noise of transistors and other devices
- weighted sum of Lorentzians:



- special filtering, e.g.:





Properties:

- limited frequency range

- both low and high frequency cutoff:

$$\sigma^2 = \int_{f_1}^{f_2} \frac{c}{f} df = c \cdot \ln\left(\frac{f_2}{f_1}\right)$$

5.1.3. Solving differential equations

<u>General recommendations:</u>

- introduce new variables
- convert to integral equations
- check stability and precision

New variables:

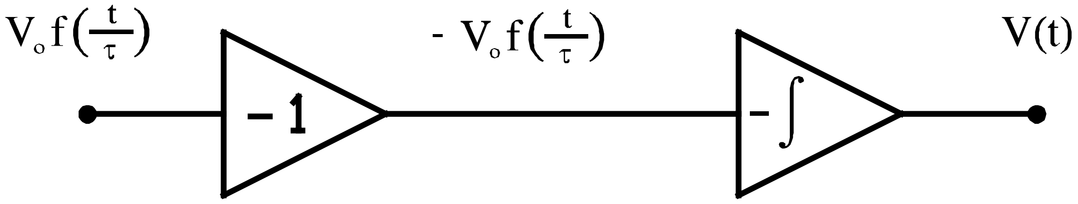
$$\left.\begin{aligned} x &= \frac{1}{\tau}\, t \\ y &= \frac{1}{V_o}\, V \end{aligned}\right\} \Rightarrow y(x) = \frac{V(t)}{V_o}$$

Example #1:

$$\frac{dy}{dx} = f(x)$$

$$y(x) = y(0) + \int_0^x f(x')\,dx'$$

$$V(t) = V(0) + \frac{1}{\tau}\int_0^t V_o f\left(\frac{t'}{\tau}\right)dt'$$

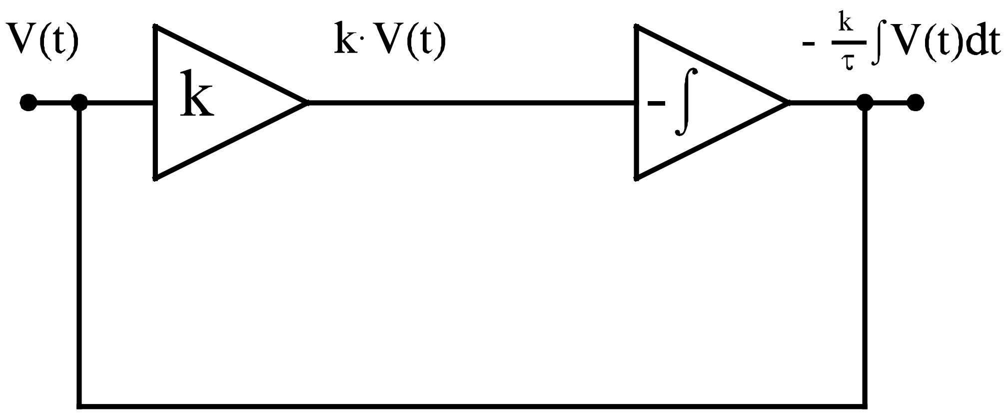


Example #2:

$$\frac{dy}{dx} = -ky$$

$$y(x) = y(0) - k\int_0^x y(x')\,dx'$$

$$V(t) = V(0) - \frac{k}{\tau}\int_0^t V(t')\,dt'$$



Note: Higher order, coupled and non-linear equations also can be solved

5.1.4. Practical considerations

- frequency range
- precision
- stability
- voltage range (truncation, noise)
- ...

5.2. Numerical simulations

Numerical simulations are very efficient:

- all advantages of digital representation
- arbitrary precision (at the expense of speed)
- only software to be "realized"
- fast improvement of hardware (computers)
- extensive, tested libraries for wide range of problems
- can be mixed with analog modellings

Some disadvantages:

- artificial systems
- software can be very complex for simple problems
- programming language limitations
- sometimes computation speed is low

5.2.1. Monte Carlo methods

- solution of matemathical and physical problems using random quantities

5.2.2. Generating random numbers

<u>Properties:</u>

- pseudo-random numbers: deterministic
- reproducable
- tests, theory always required

<u>Recommendations:</u>

- do not use built-in generators
- higher order bits are "more random"
- do not use floating point arithmetic

5.2.2.1. Linear congruential method

$$x_{n+1} = (a \cdot x_n + c) \mod m$$

Max. period: m (not neccesarrily good) if:

- c is relative prim to m
- (a-1)/p=int, for all p prim divisor of m
- (a-1)/4=int, if m/4=int

Some tested values:

a=1664525, $m=2^{32}$

a=69069, $m=2^{32}$

a=1812433253, $m=2^{32}$

BAD!!! : a=65539, $m=2^{31}$

5.2.2.2. Additive method

$$x_n = (x_{n-24} + x_{n-55}) \bmod m$$

- experimentally tested
- no proper theory
- fast, simple, long period > 2^{55}

5.2.2.3. Shuffling random numbers

Aim: improve randomness

Method: shuffling

Initial state

- x_n random numbers to be shuffled

- k fixed, $v_0,...,v_{k-1}$ filled

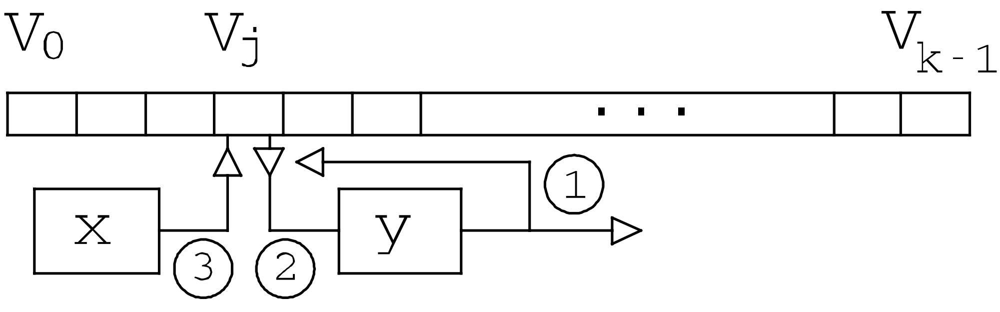
- $y=x_k$

Algorithm (one cycle):

- $j=ky/m$

- $y=v_j$

- $v_j=x_n$, next random value

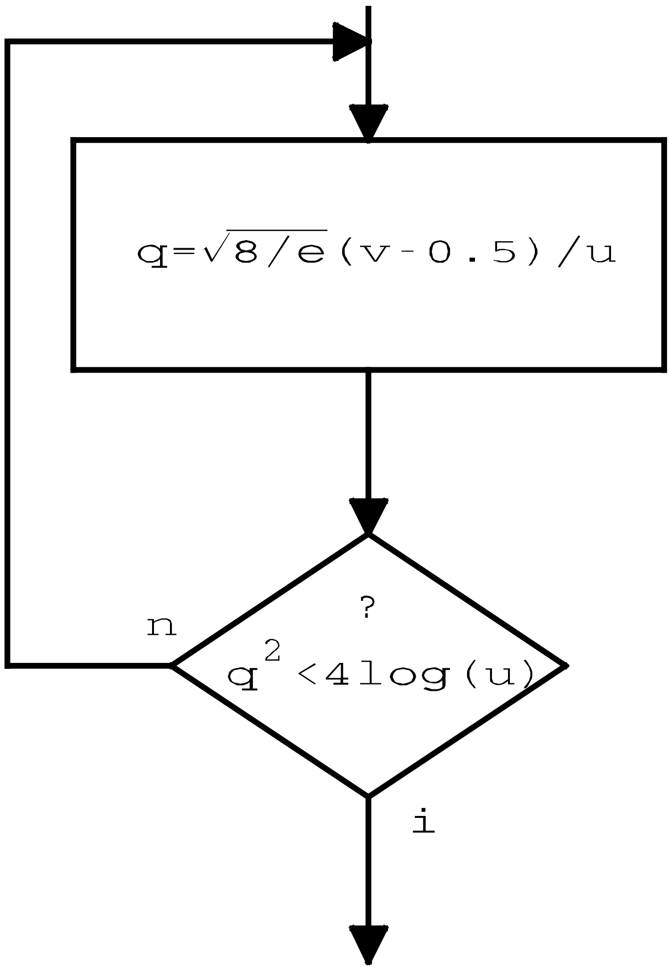


5.2.2.4. Normal distribution

Using central limit theorem

$$x=\sqrt{\frac{3}{n}}\sum_{i=1}^{n}(2y_i-1)$$

Other methods also, e.g.:



5.2.2.5. Arbitrary distribution

If y denotes a random variable with uniform distribution over 0..1, and F(x) is the desired distribution:

$$x=F^{-1}(y)$$

Sometimes it can be complex, e.g.: normal distribution

5.2.2.6. Generating noises with different spectra

White noise:

- random generators provide white noise

Other noises:

- digital implementation of analog methods (digital filters)
- FFT:

  w_i -> FFT -> $H(f) \cdot W(f)$ -> IFFT -> y_i
- special methods

Lorentzian noise:

- limited random walk
- $x_{n+1} = c \cdot x_n + w_n$, $0 < c < 1$

$1/f^2$ noise:

- random walk: $x_{n+1} = x_n + w_n$

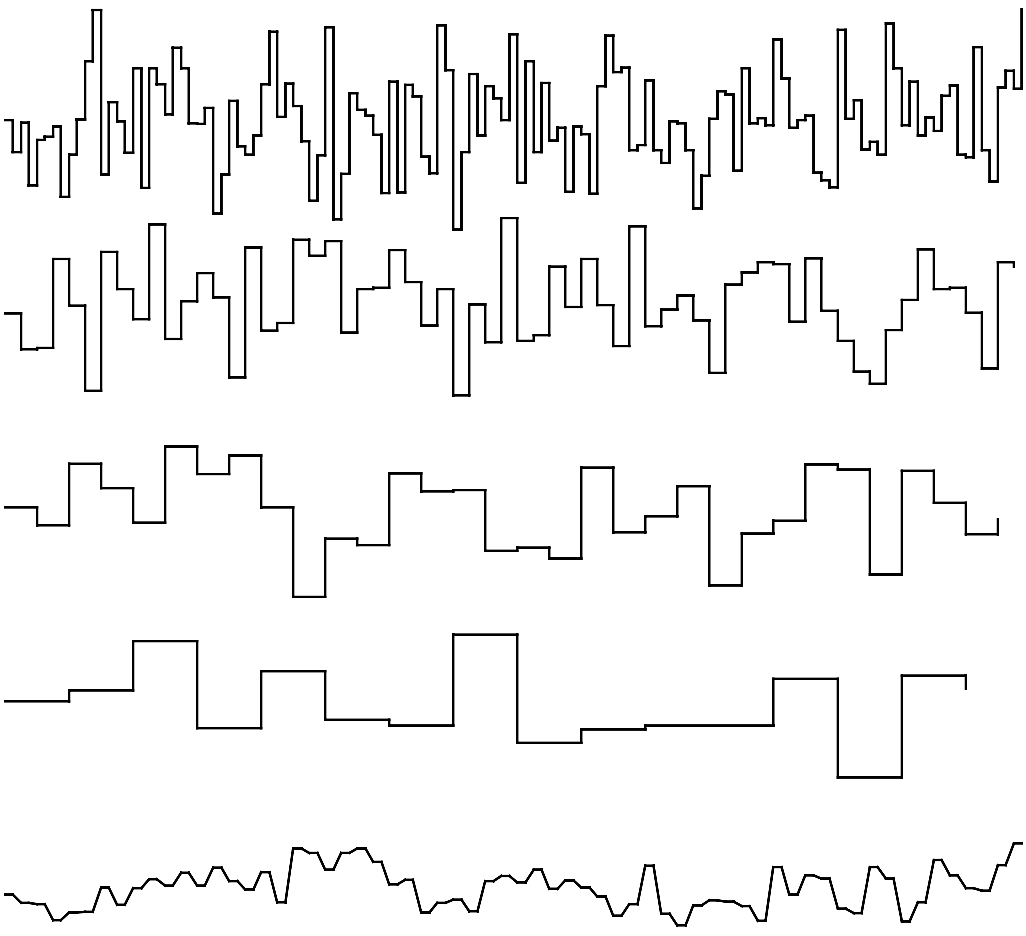
1/f noise:

- weighted sum of Lorentzians:

Example:

- Gaussian amplitude distribution

- swithcing times: $T_i = T_o \cdot 2^i$



5.2.3. Random decisions

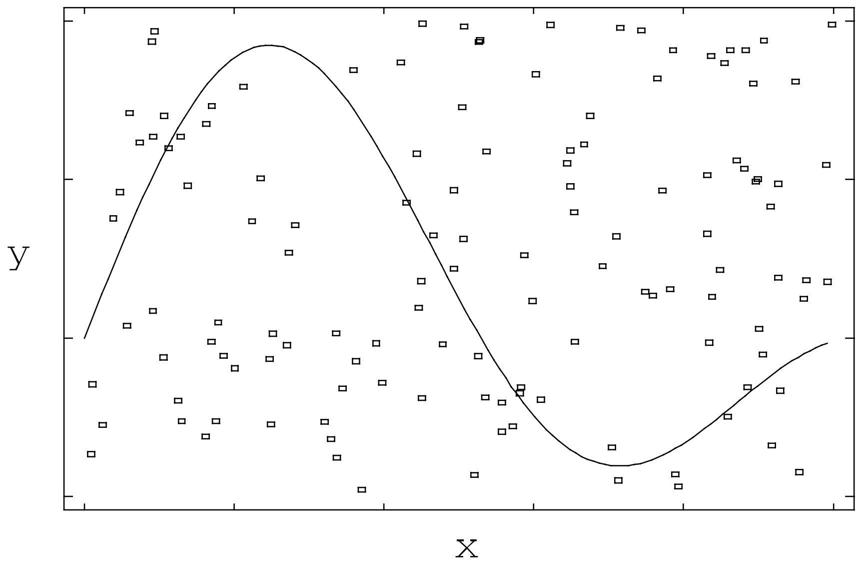
Generate a random event with probability p:

- generate a random number x in [0,1)
- if x<p then generate the event

5.2.4. Some applications of MC

MC methods have extremely wide range of applications from integration, solution of algebraic and differential equations to simulation of complex physical systems.

5.2.4.1. Integration



$$I \approx N_{under}/N_{total} * A_{rect}$$

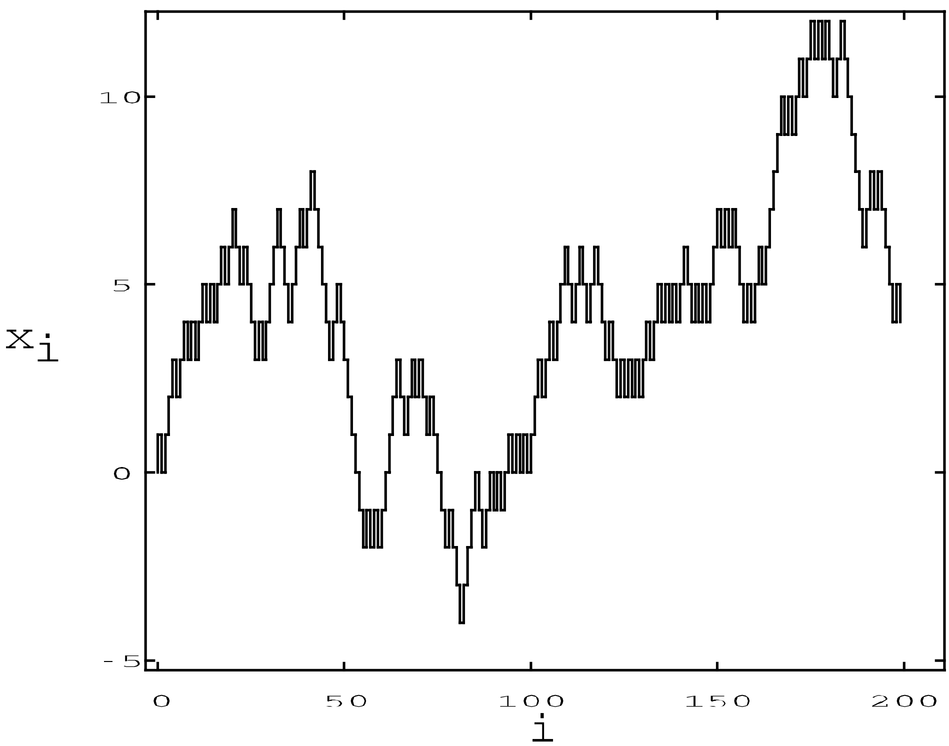
5.2.4.2. Random walk

Discrete modelling of Brownian motion:

$$x_{i+1}=x_i+w_i$$

where w_i is a random variable, +1 or -1 with probability 0.5

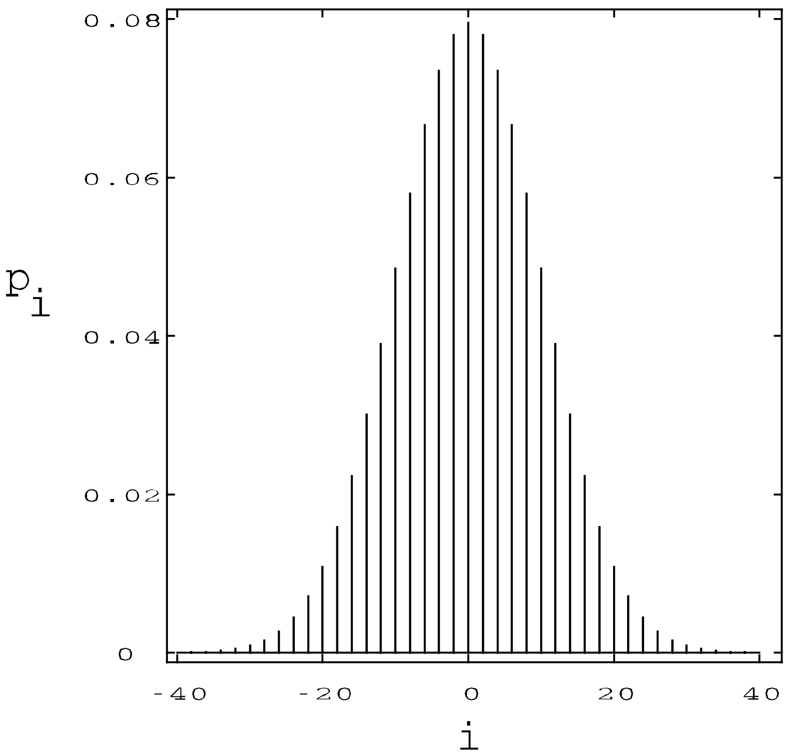
This can be considered as a numeric integration of w_i.



Distribution of amplitude after n steps:

$$p_{n,i} = \frac{1}{2^n} \binom{n}{\frac{n-i}{2}}$$

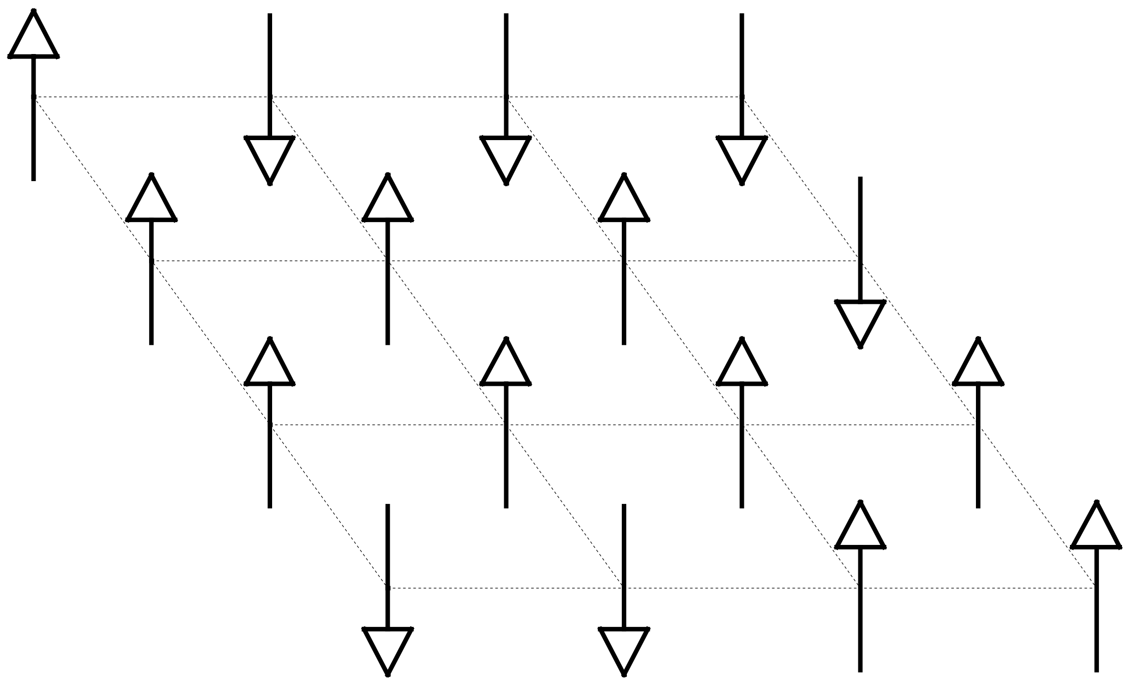
For n=100:



5.2.4.3. 2D Ising model

A simple model of magnetism: Ising model

- elementary magnets: spins
- interaction between neighbouring magnets only
- temperature -> energy for altering spins



Hamiltonian:

$$H = -J \sum_{\langle i,j \rangle} S_i S_j$$

$S_i = \pm 1$; <i,j>: sum on nearest neighbours

Monte Carlo simulation:

1. Select a set of values S_i
2. Consider a given spin S_i
3. Calculate the local energy for this spin:

$$E_i = -J \sum_{\langle i,j \rangle}^{nn} S_i S_j$$

4. S_i=-S_i

5. Calculate the new local energy:

$$E_i^{/}=-J\sum_{\langle i,j\rangle}^{nn} S_i S_j$$

6. Evaluate ΔE=E'-E

7. Evaluate the transition probability:

$$W(S_i\rightarrow -S_i)=\exp(-\Delta E/kT)$$

8. Choose a random number x: 0<x<1

9. if W(S_i->-S_i) < x, then change S_i to -S_i

10. Repeat steps 2-9 for all different spins

5.3. Mixed signal simulations

Sometimes it is advantageous/neccessarry to use analog/measured signals (e.g. noise) in modellings.

Possibilities:

- analog simulations
- using digital circuits also: mixed signal modellings

5.3.1. Digital signal processing

Mixed signal processing requires:

- sampling and reconstruction of analog signals

- knowledge of properties and limitations of digital signal processing

5.3.1.1. A/D and D/A conversion

Analog signal -> A/D -> digital signal

Digital signal -> D/A -> analog signal

5.3.1.2. Digital signal processors

Digital part of the system:

- basic logic circuits
- complex logic circuits

For more efficient processing

- computers and software
- DSPs (digital signal processors) and software

5.3.1.3. Digital noise generators

Using a random generator and a D/A converter

- simple logic circuits and D/A
- DSPs and D/A

5.3.2. Aliasing in numerical modellings

- Aliasing occurs, when the sampling frequency $f_s \leq 2 * f_{max}$, the maximum frequency in the signal
- In numerical modellings there is an f_s!

An example: 1/2 duty cycle square wave:

$$S(f) \propto \sum_{k=1}^{\infty} \frac{1}{k^2 \cdot f_o^2}$$